Heat transfer in slip flow at low Reynolds number

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The transfer of heat by forced convection from a hot wire in a low Mach number rarefied gas stream is investigated for small Reynolds numbers. A simple expression is found for the Nusselt number, and it is suggested that this has reasonable validity over a wide range of Knudsen numbers, even into the free-molecule flow régime.

1. Introduction

The hot-wire anemometer is in extensive use as a means of measuring velocity fluctuations, and for quick response the wires used are of very small diameter. Thus, in rarefied gases it is quite easy to achieve the condition where the mean free path in the gas is comparable with the wire diameter. For example, wires of 0·0001 in. diameter are often used; and while this is 40 times the mean free path in air at room conditions, it is about equal to the mean free path in air at a pressure of 2 cm of mercury and at room temperature.

While molecular effects are significant in such conditions, the Reynolds number R based on the wire diameter can be quite small, and this is the case we consider here. We are then able to use the Oseen approximation (§3), since convection terms are only important at large distances (Cole & Roshko 1954), and this effectively enables us to treat the energy equation without a knowledge of the velocity field. The model we take for the wire is an infinite cylinder of circular cross-section, so that the flow is two-dimensional; it appears likely from available measurements that this is a good approximation in forced convection for wires of aspect ratio greater than a few thousand (Collis & Williams 1957).

The significance of molecular effects is determined by the magnitude of the Knudsen number K, the ratio of the mean free path in the gas L, to the wire diameter d. When K is negligible the gas behaves as a continuum; and when it is large, intermolecular collisions are unimportant and the oncoming gas molecules are unaffected by the presence of the wire (free-molecule flow). Slip flow occurs when K is small but not negligible, and here the mean tangential gas velocity is not zero at the wire and the gas temperature at the wire differs from the wire temperature. Finally, there is a so-called transition régime between the slip-flow and free-molecule régimes about which little is known.

We work throughout with the Navier–Stokes equations. This has often been suggested, for example, by Laitone (1956), but a consideration of the Burnett terms led Lin & Street (1954) to regard them as valid if the product of K and the

Mach number M is small. Now K may also be expressed in terms of M and R, in fact

$$K = \pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}2^{-\frac{1}{2}}M/R \tag{1.1}$$

very closely, so that on these grounds the Navier–Stokes equations would be valid if M^2/R is small. We consider only the case when the density and temperature perturbations referred to their main-stream values, s and t respectively, are small; in § 2.1 we show that this again demands that M^2/R should be small, and accordingly we restrict ourselves to this condition. The present writer nevertheless believes, on closer examination of the Burnett terms, that t^2/R should also be small, which is slightly more restrictive still. Note, however, that under these conditions K may still be large.

In § 2.2 the boundary condition at the cylinder-gas interface is considered at some length, and arguments are given to support the use of a simple temperature-jump boundary condition when K ranges from zero to large values. Briefly, the reasons justifying this are its accordance with the boundary conditions associated with the Burnett equations and its *a posteriori* plausibility—in that order-of-magnitude agreement is achieved with the low Mach number free-molecule result when K is large.

The solution for the Nusselt number N_u, the non-dimensional heat transfer parameter, is carried through in § 3 without restriction on the magnitude of K. The Oseen approximation to the energy equation together with the temperature-jump boundary condition leads to an infinite set of algebraic equations for an infinite number of unknown coefficients. Their asymptotic solution for small R is obtained and leads to a simple expression for N_u, which of course reduces to the continuum result when K is zero, and for large K predicts that N_u is proportional to σ/K, where σ is the Prandtl number. This agrees with the free-molecule low Mach number result, but in numerical magnitude is out by a factor of about 1·8, and it is suggested that our result could be used as an interpolation formula between the free-molecule and continuum régimes by adjustment of the value of the 'jump' parameter. It may be added that our result shows that the continuum result would be seriously in error when $K = O(\log[8/\sigma R])$.

2.1. The governing equations

Let p, ρ, T be the pressure, density and temperature in an ideal gas which flows steadily past a body, and denote by $p_\infty, \rho_\infty, T_\infty$ their values at a large distance where the flow is uniform in the x_1 direction with speed U. We define

$$s = (\rho-\rho_\infty)/\rho_\infty, \quad t = (T-T_\infty)/T_\infty, \quad P = (p-p_\infty)/(\rho_\infty U^2), \tag{2.1}$$

and suppose that s and t are small compared with unity. If u_i are the non-dimensional velocity components referred to U, and the x_i are non-dimensional Cartesian co-ordinates referred to a typical body dimension, then the Navier–Stokes equations become, when terms of order s, t are neglected compared with terms of order unity,

$$\partial u_i/\partial x_i = 0, \tag{2.2}$$

$$u_j \partial u_i/\partial x_j = -\partial P/\partial x_i + R^{-1}\nabla^2 u_i, \tag{2.3}$$

$$u_j \partial t/\partial x_j - R^{-1}\sigma^{-1}\nabla^2 t = (\gamma-1)M^2 u_j \partial P/\partial x_j + R^{-1}(\gamma-1)M^2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2, \tag{2.4}$$

and

$$s + t = \gamma M^2 P, \tag{2.5}$$

where R is the Reynolds number referred to the typical body dimension, σ is the Prandtl number, γ is the ratio of specific heats, and M is the Mach number at infinity. It is consistent here to suppose that the thermal conductivity and the viscosity are constant. Note that (2.2) and (2.3) are the usual 'incompressible' equations which are independent of the temperature distribution, so that (2.4) is linear. The inhomogeneous terms are retained because t is not necessarily large compared with M^2.

We suppose now that R is small, so that we expect P to be of order R^{-1} roughly (for two-dimensional continuum flow past a circular cylinder it is of order $R^{-1}/\log(R^{-1})$ (Lamb 1945)); and hence if s and t are to be small then, from (2.5), M^2R^{-1} must be small. However, MR^{-1} may still be large.

2.2. The boundary conditions

We intend to use the well-known argument (Cole & Roshko 1954) that, because convection terms are small near the body and become dominant at large distances where the velocity is uniform (that is, the term $R\sigma u_j\, \partial\theta/\partial x_j$ is only important when $u_j \sim (1, 0)$), then no knowledge of the velocity field is needed to find a first approximation to solutions of the energy equation; and thus we shall only consider boundary conditions for t. (If this argument is applied when the perturbations t and s are not small, however, equation (2.4) is still non-linear due to the dissipation terms.)

From now on we shall be concerned with two-dimensional flow past a circular cylinder, and will take the cylinder radius as the typical length. However, by convention the Reynolds number will now be referred to the cylinder diameter, so that in the previous equations R is replaced by $\frac{1}{2}R$. Further, we will work in terms of plane polar co-ordinates (r, θ) with

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta. \tag{2.6}$$

As boundary conditions on t, we have

$$t \to 0 \quad \text{when} \quad r \to \infty, \tag{2.7}$$

and at the cylinder surface, $r = 1$, we will impose the temperature-jump condition

$$T_s - T_w = g\, \partial T_s/\partial r, \tag{2.8}$$

where T_s is the gas temperature at the wall, T_w is the wall temperature and g is a 'jump distance'. An explicit value for g often quoted, for example by Lin & Street (1954) and by Collis & Williams (1958), is

$$g = \frac{2-\alpha}{\alpha}\frac{4c\gamma}{\gamma+1}\frac{L}{\sigma}, \tag{2.9}$$

where L is the mean free path (evaluated at T_∞ to our order of approximation), α is the accommodation coefficient, usually in the neighbourhood of 0·9 for platinum-air interfaces, and c is a constant which depends on the molecular model chosen, but is anyway very close to 0·5. The derivation is referred back to Kennard (1938), although he took some pains to point out that T_s and $\partial T_s/\partial r$ do not refer to actual gas temperatures and their derivatives at the wall in his derivation. Nevertheless (2.8) and the equivalent of (2.9) appear as the significant part of the boundary condition for the Burnett equations (Lin & Street 1954), provided that MK and tK are small even though K is not. If account is taken of (2.5) and the remarks at the end of § 2.1, these conditions together with those quoted in the introduction for the validity of the Navier–Stokes equations would allow M to be as great as $R^{\frac{3}{4}}$, for example, and hence K to be of order $R^{-\frac{1}{4}}$.

We intend then to adopt the boundary condition (2.8) for both small and large values of K, with g given by (2.9), although perhaps the variation of g with $\sigma^{-1}L$ may be taken with less reserve than the remaining factor. Further grounds for the plausibility of this boundary condition are that it produces some aspects of free-molecule theory when K becomes large, in that the gas temperature tends to become independent of the wall temperature, and certainly the predicted Nusselt number in this case is of the right order of magnitude when compared with that given by free-molecule theory.

In terms of the non-dimensional temperature, the boundary condition becomes

$$t_{r=1} - t_w = h(\partial t/\partial r)_{r=1}, \tag{2.10}$$

$$h = \frac{2-\alpha}{\alpha}\frac{8c\gamma}{\gamma+1}\frac{K}{\sigma}. \tag{2.11}$$

3. The heat transfer

The energy equation now has the form

$$\nabla^2 t - \tfrac{1}{2}\sigma R u_i \,\partial t/\partial x_i = -\sigma M^2 \chi(x_1, x_2), \tag{3.1}$$

where χ is a known function of x_1 and x_2, and the boundary conditions are (2.7) and (2.10). Let $t^{(1)}$ be the temperature field satisfying (3.1) for which the total heat transfer from the cylinder is zero and the cylinder has the constant temperature T_e, the equilibrium temperature, and let $t^{(2)}$ be the corresponding temperature field when the cylinder is at constant temperature T_w. Then

$$t^{(3)} = t^{(2)} - t^{(1)} \tag{3.2}$$

satisfies

$$\nabla^2 t^{(3)} - 2\lambda u_i \,\partial t^{(3)}/\partial x_i = 0, \tag{3.3}$$

where

$$\lambda = \tfrac{1}{4}\sigma R, \tag{3.4}$$

and also satisfies the boundary conditions (2.7) and

$$t^{(3)}(1,\theta) - (t_w - t_e) = h(\partial t^{(3)}/\partial r)_{r=1}. \tag{3.5}$$

If Q is the total thermal flux from the cylinder, then this together with the contribution from the source field $\sigma M^2\chi$ must appear at infinity. Thus, if C is a very large circle surrounding the cylinder, Q together with the contribution from the source field must equal the rate of thermal transfer across C by conduction and convection, that is

$$\frac{Q}{kT_\infty} = -\oint_C \left(\frac{\partial t^{(2)}}{\partial r} - 2\lambda t^{(2)} u_n\right) ds - \iint \sigma M^2 \chi \, dS,$$

while

$$0 = -\oint_C \left(\frac{\partial t^{(1)}}{\partial r} - 2\lambda t^{(1)} u_n\right) ds - \iint \sigma M^2 \chi \, dS,$$

where k is the thermal conductivity and u_n is the outward normal velocity component on C.

Thus
$$\frac{Q}{kT_\infty} = -\oint_C \left(\frac{\partial t^{(3)}}{\partial r} - 2\lambda t^{(3)} u_n\right) ds.$$

Of course, it is obvious from these expressions that equation (3.1) commits us to take the thermal flux from the cylinder as $-k(\partial T/\partial r)_{r=1}$.

Let us finally change the dependent variable to
$$\phi = t^{(3)}/(t_w - t_e), \tag{3.6}$$
and then with the Oseen approximation we have
$$\nabla^2\phi - 2\lambda\, \partial\phi/\partial x_1 = 0, \tag{3.7}$$
$$\phi(1,\theta) - 1 = h(\partial\phi/\partial r)_{r=1}, \tag{3.8}$$
and
$$\phi \to 0 \quad \text{as} \quad r \to \infty, \tag{3.9}$$
while
$$N_u = \frac{Q}{\pi k(T_w - T_e)} = -\frac{1}{\pi}\oint_C \left(\frac{\partial\phi}{\partial r} - 2\lambda\phi\cos\theta\right) ds, \tag{3.10}$$
where N_u is the Nusselt number.

We follow Cole & Roshko (1954) and let
$$\phi = \psi \exp(\lambda r\cos\theta), \tag{3.11}$$
so that in terms of ψ
$$\nabla^2\psi - \lambda^2\psi = 0, \tag{3.12}$$
with boundary conditions
$$\psi \exp(\lambda r\cos\theta) \to 0 \quad \text{when} \quad (r \to \infty) \tag{3.13}$$
and
$$(1 - h\lambda\cos\theta)\,\psi_{r=1} - \exp(-\lambda\cos\theta) = h(\partial\psi/\partial r)_{r=1}. \tag{3.14}$$
Since ψ must be symmetrical about the x_1 axis, the appropriate solution which meets the boundary condition of infinity is of the form
$$\psi = \sum_0^\infty A_n K_n(\lambda r)\cos n\theta, \tag{3.15}$$
and from (3.10)
$$N_u = \lim_{r\to\infty} \frac{\lambda r}{\pi}\int_0^{2\pi} \exp(\lambda r\cos\theta) \sum_{n=0}^\infty A_n\{K_n(\lambda r)\cos n\theta\cos\theta - K_n'(\lambda r)\cos n\theta\}\, d\theta.$$
If we use the relation
$$2\pi I_n(z) = \int_0^{2\pi} \cos n\theta \exp(z\cos\theta)\, d\theta \tag{3.16}$$
and the known asymptotic forms for $I_n(z)$ and $K_n(z)$, this reduces to
$$N_u = 2\sum_0^\infty A_n. \tag{3.17}$$

It remains now to satisfy the boundary condition (3.14) at the cylinder wall, and since
$$\exp(-\lambda\cos\theta) = I_0(\lambda) + 2\sum_{n=1}^\infty (-1)^n I_n(\lambda)\cos n\theta,$$
we arrive at the set of equations
$$\left.\begin{aligned}
[K_0(\lambda) - h\lambda K_0'(\lambda)]A_0 - \tfrac{1}{2}h\lambda K_1(\lambda)A_1 &= I_0(\lambda),\\
-h\lambda K_0(\lambda)A_0 + [K_1(\lambda) - h\lambda K_1'(\lambda)]A_1 - \tfrac{1}{2}h\lambda K_2(\lambda)A_2 &= -2I_1(\lambda),\\
-\tfrac{1}{2}h\lambda K_1(\lambda)A_1 + [K_2(\lambda) - h\lambda K_2'(\lambda)]A_2 - \tfrac{1}{2}h\lambda K_3(\lambda)A_3 &= 2I_2(\lambda),\\
\textit{etc.}
\end{aligned}\right\} \tag{3.18}$$

We propose to discuss the asymptotic solution of this infinite set of equations for small λ by considering the first N, say, where N is large, and then allowing N to become infinite.

The determinant Δ of the finite set is a continuant whose diagonal terms, super-diagonal terms and subdiagonal terms are now denoted by a_n, b_n, c_n, respectively, where

$$\left.\begin{aligned} a_n &= K_{n-1}(\lambda)-h\lambda K'_{n-1}(\lambda) \quad (n \geqslant 1),\\ b_1 &= h\lambda K_1(\lambda),\\ b_n &= -\tfrac{1}{2}h\lambda K_n(\lambda), \quad (n \geqslant 2),\\ c_1 &= -h\lambda K_0(\lambda),\\ c_n &= -\tfrac{1}{2}h\lambda K_{n-1}(\lambda), \quad (n \geqslant 2).\end{aligned}\right\} \tag{3.19}$$

and

Then by Cramér's rule we have

$$A_m \Delta = D_{m+1}, \tag{3.20}$$

where D_n is the determinant obtained by replacing the nth column of Δ by the column with elements d_n defined by

$$\left.\begin{aligned} d_1 &= I_0(\lambda),\\ d_n &= (-1)^n\, 2I_{n-1}(\lambda) \quad (n \geqslant 2).\end{aligned}\right\} \tag{3.21}$$

Now, since Δ is a continuant, it is easy to show that if a non-diagonal element, say b_n, occurs in a term of its expansion, then c_n must also occur. If we then take account of the behaviour of $I_n(\lambda)$ and $K_n(\lambda)$ for small λ, it is possible to show that the dominant term in the expansion of Δ is given by the product of the diagonal elements, and in fact

$$\Delta = \left(\prod_{m=1}^{N} a_m\right)(1+\epsilon), \tag{3.22}$$

where

$$\left.\begin{aligned} \epsilon &= O(\lambda^2 \log \lambda) \quad \text{if} \quad h^{-1} = o(1),\\ &= O(\lambda^2) \quad \text{if} \quad h = O(1),\\ &= O(h^2\lambda^2) \quad \text{if} \quad h = o(1).\end{aligned}\right\} \tag{3.23}$$

A discussion of the determinants D_n proceeds similarly although with rather more complication, and we can show that

$$D_1 = d_1\left(\prod_{m=2}^{N} a_m\right)(1+\epsilon), \tag{3.24}$$

and

$$D_n/D_1 = O(\lambda^{2n-2}\log\lambda) \quad (n \geqslant 2), \tag{3.25}$$

for all h.

Hence

$$a_1 A_0 = d_1(1+\epsilon), \tag{3.26}$$

and

$$\sum_{n=0}^{N} A_n = A_0(1+\epsilon). \tag{3.27}$$

Thus, finally, for small λ and uniformly for all h,

$$\begin{aligned} N_u = 2\sum_{n=0}^{\infty} A_n &\sim 2d_1/a_1\\ &\sim 2(h-\Gamma+\log(2/\lambda))^{-1}, \end{aligned} \tag{3.28}$$

where Γ ($\doteqdot$ 0·5772) is Euler's constant. In terms of the Reynolds number this result may be written

$$N_u^{-1} = \tfrac{1}{2}h - \tfrac{1}{2}\Gamma + \tfrac{1}{2}\log(8\sigma^{-1}R^{-1}) = \tfrac{1}{2}h + N_{uc}^{-1}, \tag{3.29}$$

where N_{uc} is the continuum result of Cole & Roshko.

When h is large,

$$N_u \sim 2h^{-1} = \frac{\gamma+1}{4c\gamma}\frac{\alpha}{2-\alpha}\frac{\sigma}{K}. \tag{3.30}$$

The low Mach number free-molecule result for a diatomic gas is (Stalder, Goodwin & Creager 1951)

$$N_u \sim \frac{3}{2}\frac{\gamma-1}{\gamma}\alpha\frac{\sigma}{K}, \tag{3.31}$$

so that although (3.30) predicts the correct variation with σ and K the accompanying factor is too large by a factor of about 1·8. However, in view of the assumptions made, this is quite reasonable, and it is plausible that (3.29) would provide a reasonable interpolation formula between the free-molecule régime and continuum flow if h was redefined to make (3.30) and (3.31) agree. For there are only *strong* grounds for accepting the original definition of h, i.e. equation (2.11), in a range of K where the continuum result is not greatly affected.

It is interesting to note that on other grounds Collis & Williams (1958) have attempted to fit an empirical expression of similar type to available experimental results for larger Reynolds numbers than those considered here, and have to take the rather unrealistic value of 0·695 for α in (2.11) (which reduces the above discrepancy to a factor of 1·18).

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